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Superfields with internal symmetry

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Abstract. The generalization of superfields to include an internal SU(n) symmetry group is studied, and shown to be a simple extension of the original formulation of Ferrara, Wess and Zumino.

1. Introduction

One of the more interesting, and perhaps physically significant extensions of the original Wess and Zumino (1974a, b) supersymmetry is to combine it with an internal symmetry group to obtain a new defining algebra (Salam and Strathdee 1974b, Dondi and Sohnius 1974). This can be effected by allowing the charges which generate the supersymmetry (and obey anticommutation relations rather than commutation relations) to become non-trivial representations of the internal symmetry group.

By considering the algebra of these charges together with the generators of the Poincaré group, it is possible to obtain representations of the combined supersymmetry and internal symmetry. In order to find these, Salam and Strathdee (1974b), Wess (1974) and Zumino (1974) have shown how to construct rest-frame representations in terms of massive particle states. Applying Lorentz boosts to these states gives basis vectors for the representations of the complete algebra. It is also possible to approach the problem of supersymmetry representations by means of the superfield technique of Salam and Strathdee (1974a), and this has been demonstrated for the internal symmetry group SU(2) (Dondi and Sohnius 1974). The superfield can then be used to construct Lagrangian models invariant under the combined supersymmetry and internal SU(2) group (Wess 1974, Firth and Jenkins 1975, Dondi and Sohnius, unpublished, Capper and Leibbrandt 1975, Dondi and Wess, unpublished).

Because the two symmetries are mixed non-trivially, the number of independent parameters on which the superfield depends increases with the dimension of the internal symmetry and the structure of the superfield becomes more and more complicated, as can easily be seen by glancing at Dondi and Sohnius (1974) and Ferrara *et al* (1974). The aim of this paper is to provide techniques which apply generally to SU(n) symmetries and lead to relatively simple power series expansions for the superfields in terms of their ordinary field components. In § 2, we redevelop the algebra and superfield transformations to include SU(n) as the internal symmetry group. The power series expansions of some simple superfields are considered in § 3, and in § 4 we discuss some specific examples of Lagrangian models with SU(2) symmetry to demonstrate the power of our methods.

2. The algebra and transformation laws

The supersymmetry generators consist of the $2n$ component complex spinor Q^j_α, transforming as a two-dimensional spinor representation of the Lorentz group (Greek index) and a basic n-dimensional representation of SU(n) (Latin index) together with its Hermitian conjugate $\bar{Q}_{\dot{\alpha}j}$. The defining commutators can be written as

$$[M_{\mu\nu}, Q^j_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^j_\beta \qquad \text{sum over repeated indices} \tag{1a}$$

$$[M_{\mu\nu}, \bar{Q}_{j\dot{\alpha}}] = \tfrac{1}{2}\bar{Q}_{j\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{1b}$$

$$[F_P, Q^j_\alpha] = -\tfrac{1}{2}(\lambda_P)^j{}_k Q^k{}_\alpha \tag{2a}$$

$$[F_P, \bar{Q}_{j\dot{\alpha}}] = \tfrac{1}{2}\bar{Q}_{k\dot{\alpha}}(\lambda_P)^k{}_j \tag{2b}$$

where $M_{\mu\nu}$ are the Lorentz group generators†, F_P generate the internal SU(n) symmetry group, the matrices λ_P are the basic representation of SU(n), and $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ are two, two-dimensional matrix representations of the Lorentz group (see appendix).

In order to simplify the following calculations, it is convenient to replace the SU(n) and SL(2, C) labels by SU($2n$) indices. Thus, spinors are redefined in the following manner:

$$u^j{}_\alpha \equiv u^A \qquad \text{and} \qquad v^\alpha_j \equiv v_A$$

whilst their Hermitian conjugates are

$$\bar{u}_{j\dot{\alpha}} \equiv \bar{u}_{\dot{A}} \qquad \text{and} \qquad \bar{v}^{\dot{\alpha}j} \equiv \bar{v}^{\dot{A}}$$

and the usual rules of spinor analysis apply, such that contractions are made by summing over dotted or undotted pairs of indices, one covariant and one contravariant.

Then we can replace equations (1) and (2) by

$$[M_{\mu\nu}, Q^A] = -\tfrac{1}{2}(\Sigma_{\mu\nu})^A{}_B Q^B \tag{3a}$$

$$[M_{\mu\nu}, \bar{Q}_{\dot{A}}] = \tfrac{1}{2}\bar{Q}_{\dot{B}}(\bar{\Sigma}_{\mu\nu})^{\dot{B}}{}_{\dot{A}} \tag{3b}$$

$$[F_P, Q^A] = -\tfrac{1}{2}(\Lambda_P)^A{}_B Q^B \tag{4a}$$

$$[F_P, \bar{Q}_{\dot{A}}] = \tfrac{1}{2}\bar{Q}_{\dot{B}}(\Lambda_P)^{\dot{B}}{}_{\dot{A}} \tag{4b}$$

where we have also combined the matrix representation indices in the form

$$(\Sigma_{\mu\nu})^A{}_B = \delta^j{}_k(\sigma_{\mu\nu})_\alpha{}^\beta$$

$$(\bar{\Sigma}_{\mu\nu})^{\dot{B}}{}_{\dot{A}} = \delta^k{}_j(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}$$

$$(\Lambda_P)^A{}_B = (\lambda_P)^j{}_k\,\delta_\alpha{}^\beta$$

$$(\Lambda_P)^{\dot{B}}{}_{\dot{A}} = (\lambda_P)^k{}_j\,\delta^{\dot{\beta}}{}_{\dot{\alpha}}.$$

To complete the defining algebra we take the algebra of the Poincaré group generators, $M_{\mu\nu}$ and P_μ together with the SU(n) algebra, and the remaining relations between the spinor charges and P_μ given by

$$\{Q^A, Q^B\} = \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = 0 \tag{5a}$$

$$\{Q^A, \bar{Q}_{\dot{B}}\} = 2(\sigma_\mu)^A{}_{\dot{B}} P^\mu \tag{5b}$$

$$[P_\mu, Q^A] = [P_\mu, \bar{Q}_{\dot{B}}] = 0 \tag{6}$$

† The indices $\mu, \nu = 0, 1, 2, 3$, and the metric is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

where

$$(\sigma_\mu)^A{}_{\dot{B}} = \delta^i{}_j(\sigma_\mu)_{\alpha\dot{\beta}}$$

and the σ_μ are given in the appendix.

The algebra thus defined is a generalization of the subalgebra originally introduced (Wess and Zumino 1974a, b, Salam and Strathdee 1974a, b) to allow massive particle theories to be developed.

Continuing with the now accepted method (Salam and Strathdee 1974a, Ferrara *et al* 1974), we introduce totally anticommuting parameters, which anticommute amongst themselves and also anticommute with all other spinorial quantities, including the Q^A and $\bar{Q}_{\dot{A}}$. With such a set θ_A and $\bar{\theta}^{\dot{B}}$, we can replace the anticommutators of equation (5) by the commutator algebra

$$[\theta_A Q^A, \theta_B Q^B] = [\bar{Q}_{\dot{A}}\bar{\theta}^{\dot{A}}, \bar{Q}_{\dot{B}}\bar{\theta}^{\dot{B}}] = 0 \tag{7a}$$

$$[\theta_A Q^A, \bar{Q}_{\dot{B}}\bar{\theta}^{\dot{B}}] = 2\theta_A(\sigma_\mu)^A{}_{\dot{B}}\bar{\theta}^{\dot{B}}P^\mu. \tag{7b}$$

In complete analogy with the work of Ferrara *et al* (1974), we can now define functions of the θ_A, $\bar{\theta}^{\dot{B}}$ and x_μ by†

$$\phi(x, \theta, \bar{\theta}) = \exp(-\mathrm{i}xP + \mathrm{i}\theta Q + \mathrm{i}\bar{Q}\bar{\theta}) \tag{8a}$$

$$\phi_1(x, \theta, \bar{\theta}) = \exp(-\mathrm{i}xP + \mathrm{i}\theta Q)\exp(\mathrm{i}\bar{Q}\bar{\theta}) \tag{8b}$$

$$\phi_2(x, \theta, \bar{\theta}) = \exp(-\mathrm{i}xP + \mathrm{i}\bar{Q}\bar{\theta})\exp(\mathrm{i}\theta Q) \tag{8c}$$

which are connected, using equations (6) and (7), by the shift operation

$$\phi(x_\mu, \theta, \bar{\theta}) = \phi_1(x_\mu + \mathrm{i}\theta\sigma_\mu\bar{\theta}, \theta, \bar{\theta}) = \phi_2(x_\mu - \mathrm{i}\theta\sigma_\mu\bar{\theta}, \theta, \bar{\theta}). \tag{9}$$

It is easy to see that multiplication from the left by an element of the form

$$G = \exp(\mathrm{i}\bar{Q}\bar{\eta} + \mathrm{i}\eta Q)$$

leads infinitesimally to the group action

$$\delta\phi = (\eta\partial_\theta + \bar{\eta}\partial_{\bar{\theta}} + \mathrm{i}\theta\sigma_\mu\bar{\eta}\partial^\mu - \mathrm{i}\eta\sigma_\mu\bar{\theta}\partial^\mu)\phi \tag{10a}$$

$$\delta\phi_1 = (\eta\partial_\theta + \bar{\eta}\partial_{\bar{\theta}} + 2\mathrm{i}\theta\sigma_\mu\bar{\eta}\partial^\mu)\phi_1 \tag{10b}$$

$$\delta\phi_2 = (\eta\partial_\theta + \bar{\eta}\partial_{\bar{\theta}} - 2\mathrm{i}\eta\sigma_\mu\bar{\theta}\partial^\mu)\phi_2 \tag{10c}$$

where $\eta\partial_\theta = \eta_A\partial/\partial\theta_A$.

The structure of the supersymmetry algebra and the transformations are obviously the same as in the case of no internal symmetry, and we can consider equations (9) and (10) to be basic properties of superfields (Ferrara *et al* 1974). Furthermore, it is also easy to see that sets of covariant derivatives (ie derivative operators that commute with the variations of equation (10)) can be constructed and are of the same form as their simpler counterparts without internal symmetry; the covariant derivatives are

on ϕ:

$$\mathrm{D}^A = \frac{\partial}{\partial\theta_A} + \mathrm{i}(\sigma^\mu)^A{}_{\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu \quad \text{and} \quad \bar{\mathrm{D}}_{\dot{A}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} - \mathrm{i}\theta_B(\sigma^\mu)^B{}_{\dot{A}}\partial_\mu \tag{11a}$$

† Henceforth we leave out the indices when the summations are obvious, ie

$$xP = x_\mu P^\mu, \qquad \theta Q = \theta_A Q^A.$$

on ϕ_1:

$$\mathrm{D}^A = \frac{\partial}{\partial\theta_A} + 2\mathrm{i}(\sigma^\mu)^A{}_{\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu \quad \text{and} \quad \bar{\mathrm{D}}_{\dot{A}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} \tag{11b}$$

on ϕ_2:

$$\mathrm{D}^A = \frac{\partial}{\partial\theta_A} \quad \text{and} \quad \bar{\mathrm{D}}_{\dot{A}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} - 2\mathrm{i}\theta_B(\sigma^\mu)^B{}_{\dot{A}}\partial_\mu. \tag{11c}$$

The D^A and $\bar{\mathrm{D}}_{\dot{A}}$ obey the relations

$$\{\mathrm{D}^A, \mathrm{D}^B\} = \{\bar{\mathrm{D}}_{\dot{A}}, \bar{\mathrm{D}}_{\dot{B}}\} = 0 \tag{12a}$$

and

$$\{\mathrm{D}^A, \bar{\mathrm{D}}_{\dot{A}}\} = -2\mathrm{i}(\sigma_\mu)^A{}_{\dot{A}}\partial^\mu \tag{12b}$$

or equivalently

$$[\xi_A\mathrm{D}^A, \bar{\mathrm{D}}_{\dot{A}}\bar{\xi}^{\dot{A}}] = -2\mathrm{i}\xi\sigma_\mu\bar{\xi}\partial^\mu \tag{12c}$$

using the anticommuting parameter ξ_A and its Hermitian conjugate.

Thus, using this SU($2n$) notation, the algebra, transformation laws and covariant derivatives can easily be written in forms completely equivalent to those of the original supersymmetry model.

3. Superfields

The superfield $\phi(x, \theta, \bar{\theta})$ can be expanded as a finite power series in the anticommuting parameters θ and $\bar{\theta}$, with coefficients which are ordinary fields depending on x_μ and being representations of SU(n) and the Lorentz group. In general, $\phi(x, \theta, \bar{\theta})$ is not irreducible and it is possible to impose constraints, using the covariant derivatives, which reduce the number of ordinary fields in the expansion. The simplest types of constraint are of the form $\mathrm{D}^A\phi = 0$ which implies that ϕ_2 is independent of θ and $\bar{\mathrm{D}}_{\dot{A}}\phi = 0$ which gives ϕ_1 independent of $\bar{\theta}$. We will concentrate on this type of superfield.

Thus, the $\phi_1(x, \theta)$ can be expanded as

$$\phi_1(x, \theta) = A(x) + \theta_A\psi^A(x) + \frac{1}{2!}\theta_A\theta_B B^{[AB]}(x) + \frac{1}{3!}\theta_A\theta_B\theta_C v^{[ABC]}(x) + \dots + \frac{1}{(2n)!}\theta_{A_1}\theta_{A_2}\dots\theta_{A_{2n}}F^{[A_1A_2\dots A_{2n}]}(x) \tag{13}$$

where the $[ABC\dots]$ indicates total antisymmetry in these indices, and because of the $\bar{\theta}$ independence we have no dotted indices to consider in this case. The beauty of combining the SL(2, C) index with the SU(n) index is now even more apparent. We know not only that each term is an irreducible representation of SU($2n$), but also that after the $(n+1)$ term, the SU($2n$) representations that appear are simply the conjugates of those that come in the first n terms (a transparent way to see this is to consider the Young tableaux for the antisymmetric combinations of SU($2n$) basic spinors).

Remembering that in SU(n), we have at our disposal the totally antisymmetric Levi-Civita tensors $\epsilon^{A_1A_2\dots A_n}$ and $\epsilon_{A_1A_2\dots A_n}$ (both defined to have $\epsilon_{123\dots n} = \epsilon^{123\dots n} = 1$)

to raise or lower indices, we can rewrite our superfield as

$$\phi_1(x,\theta) = A(x)+\theta_A\psi^A(x)+\frac{1}{2!}\theta_A\theta_B B^{[AB]}(x)+\ldots+\frac{1}{(2n-2)!}\theta_{A_1}\theta_{A_2}\ldots\theta_{A_{2n-2}}$$

$$\times\epsilon^{A_1A_2\ldots A_{2n}}G_{[A_{2n-1}A_{2n}]}(x)+\frac{1}{(2n-1)!}\theta_{A_1}\ldots\theta_{A_{2n-1}}\epsilon^{A_1\ldots A_{2n}}\lambda_{A_{2n}}$$

$$+\frac{1}{(2n)!}\theta_{A_1}\ldots\theta_{A_{2n}}\epsilon^{A_1\ldots A_{2n}}F(x) \tag{14}$$

where we have explicitly rewritten the last few terms of the power series using the ϵ tensor. This is always possible and leads again to a structure that is an obvious generalization of the Ferrara, Wess and Zumino superfield. As particular examples, we have, for the case of no internal symmetry

$$\phi_1(x,\theta) = A(x)+\theta_A\psi^A(x)+\tfrac{1}{2}\theta_A\theta_B\epsilon^{AB}F(x) \tag{15a}$$

as given in Ferrara *et al* (1974). With SU(2) as the internal symmetry group, we find (Dondi and Sohnius 1974)

$$\phi_1(x,\theta) = A(x)+\theta_A\psi^A(x)+\frac{1}{2!}\theta_A\theta_B B^{[AB]}(x)+\frac{1}{3!}\theta_A\theta_B\theta_C\epsilon^{ABCD}\lambda_D(x)+\frac{1}{4!}\theta_A\theta_B\theta_C\theta_D\epsilon^{ABCD}F(x) \tag{15b}$$

containing the SU(4) representations

$$\underline{1}+\underline{4}+\underline{6}+\underline{\bar{4}}+\underline{1}.$$

And finally, for SU(3), the superfield is

$$\phi_1(x,\theta) = A(x)+\theta_A\psi^A(x)+\frac{1}{2!}\theta_A\theta_B B^{[AB]}(x)+\frac{1}{3!}\theta_A\theta_B\theta_C v^{[ABC]}(x)$$

$$+\frac{1}{4!}\theta_A\theta_B\theta_C\theta_D\epsilon^{ABCDEF}G_{[EF]}(x)+\frac{1}{5!}\theta_A\theta_B\theta_C\theta_D\theta_E\epsilon^{ABCDEF}\lambda_F(x)$$

$$+\frac{1}{6!}\theta_A\theta_B\theta_C\theta_D\theta_E\theta_F\epsilon^{ABCDEF}F(x) \tag{15c}$$

with SU(6) representations

$$\underline{1}+\underline{6}+\underline{15}+\underline{20}+\underline{\overline{15}}+\underline{\bar{6}}+\underline{1}.$$

From equations (14) and (15), the general structure for any SU(n) symmetry is now clear, and the transformation properties of the ordinary fields which are the coefficients of θ follow immediately by applying equation (10*b*) and equating products of θ to obtain, for example from equation (14)

$$\begin{aligned}
\delta A(x) &= \eta_A\psi^A(x)\\
\delta\psi^A(x) &= \eta_B B^{[AB]}(x)+2i(\eth\bar{\eta})^A A(x)\\
&\vdots\\
\delta\lambda_A(x) &= \eta_A F(x)+4i(\eth\bar{\eta})^B G_{[BA]}(x)\\
\delta F(x) &= 2i(\eth\bar{\eta})^A\lambda_A(x)
\end{aligned} \tag{16}$$

where

$$(\eth)^{A}{}_{\dot{B}} = (\sigma_{\mu})^{A}{}_{\dot{B}}\partial^{\mu}.$$

The basis given in equation (14) for the expansion of $\phi_1(x, \theta)$ is the most succinct way of presenting the superfield transformations. It keeps all the fields as irreducible representations of SU($2n$) and treats the Lorentz group and SU(n) on the same footing. Of course, the SU(n) $\otimes$ Lorentz group content of the SU($2n$) multiplets can easily be resurrected by general techniques of reduction. However, most manipulations such as multiplication of superfields of the same type, or shifting of superfields from one type to another, benefit from being done in the SU($2n$) notation.

Before giving examples of these manipulations, we should note a few further properties of interest in later calculations. The product of $2n$ spinor components can be written as

$$\theta_{A_1}\theta_{A_2}\ldots\theta_{A_{2n}} = \epsilon_{A_1A_2\ldots A_{2n}}u \tag{17a}$$

where

$$u = \frac{1}{2n!}\epsilon^{A_1A_2\ldots A_{2n}}\theta_{A_1}\theta_{A_2}\ldots\theta_{A_{2n}}$$

and of $2n-1$ components as

$$\theta_{A_2}\theta_{A_3}\ldots\theta_{A_{2n}} = \epsilon_{A_1A_2\ldots A_{2n}}\chi^{A_1} \tag{17b}$$

where

$$\chi^{A_1} = \frac{1}{(2n-1)!}\epsilon^{A_1A_2A_3\ldots A_{2n}}\theta_{A_2}\theta_{A_3}\ldots\theta_{A_{2n}}$$

such that

$$\theta_A\chi^B = \delta_A{}^B u.$$

We see directly that for two sets of anticommuting parameters θ and θ',

$$u(\theta)f(\theta'+\theta) = u(\theta)f(\theta') \tag{18}$$

where we have explicitly given the u as a function of θ, and $f(\theta)$ is a function with a decomposition as given in equation (14). Now if we define a derivative operator

$$\frac{1}{(2n)!}\epsilon_{B_1B_2\ldots B_{2n}}\frac{\partial}{\partial\theta_{B_{2n}}}\frac{\partial}{\partial\theta_{B_{2n-1}}}\ldots\frac{\partial}{\partial\theta_{B_1}} \equiv \frac{\mathrm{d}}{\mathrm{d}\theta^{(2n)}}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2n)}}u(\theta) = \frac{1}{[(2n)!]^2}\epsilon_{B_1B_2\ldots B_{2n}}\epsilon^{A_1A_2\ldots A_{2n}}\frac{\partial}{\partial\theta_{B_{2n}}}\ldots\frac{\partial}{\partial\theta_{B_1}}(\theta_{A_1}\ldots\theta_{A_{2n}}),$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2n)}}u(\theta) = 1. \tag{19}$$

Using equation (19), we find immediately that

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2n)}}u(\theta'-\theta)f(\theta) = f(\theta'). \tag{20}$$

Equations (18), (19) and (20) suggest a comparison of $u(\theta)$ with a δ function as has previously been noted by Fujikawa and Lang (1975) for the case $n = 1$.

4. Superfield techniques in SU(2) and Lagrangian models

For no internal symmetry, the superfield methods and several Lagrangian models have been thoroughly investigated (Fujikawa and Lang 1975, and references therein). The simplest extension to include SU(2) has several new features. Firstly, the superfield given in equation (15*b*) is not completely irreducible, and it is possible to find a further non-trivial constraint to limit the number of constituent fields (Wess 1974, Firth and Jenkins 1975, Dondi and Sohnius, unpublished). The new superfield can be used to construct invariant Lagrangian models for massless particles.

Secondly, Lagrangian models with mass can be constructed (Capper and Leibbrandt 1975, Dondi and Wess, unpublished), but are not trivial generalizations of those obtained in the simple case. Attempts to use the superfield of equation (15*b*) lead to a Lagrangian which is not manifestly free of ghosts, although Capper and Leibbrandt (1975) have shown by explicit calculation that ghosts do not appear.

In this section, we use the results of the earlier sections to study the superfield reduction for SU(2) as an internal symmetry group, and construct manifestly ghost-free Lagrangians.

To consider the further constraint on the superfield given by equation (15*b*) we need explicit expressions for the Hermitian conjugate of $\phi_1(x, \theta)$ and also the shift operation applied to $\phi_1(x, \theta)$. The Hermitian conjugate is given by

$$\phi_1^+(x, \bar{\theta}) = A^+ + \bar{\psi}_{\dot{A}}\bar{\theta}^{\dot{A}} + \tfrac{1}{2}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{A}}B^+_{[\dot{A}\dot{B}]} - \bar{\lambda}^{\dot{D}}\bar{\chi}_{\dot{D}} + \bar{u}F^+ \tag{21}$$

where we leave the x dependence of the ordinary fields implicit. This superfield is now a function of $\bar{\theta}$ only, with

$$\bar{\chi}_{\dot{D}} = -\frac{1}{3!}\epsilon_{\dot{D}\dot{A}\dot{B}\dot{C}}\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}, \qquad \bar{u} = \frac{1}{4!}\epsilon_{\dot{A}\dot{B}\dot{C}\dot{D}}\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{D}}$$

and obeys the condition $D^A\phi = 0$.

The shift operation of equation (9) leads to

$$\phi_2(x_\mu, \theta, \bar{\theta}) = \phi_1(x_\mu + 2i\theta\sigma_\mu\bar{\theta}, \theta, \bar{\theta}) = \exp(2i\theta\eth\bar{\theta})\phi_1(x, \theta) \tag{22}$$

which, expanding the exponential and using the properties of the ϵ tensor, is just

$$\phi_2(x, \theta, \bar{\theta}) = [1 + 2i(\theta\eth\bar{\theta}) - 2(\theta\eth\bar{\theta})^2 + 8i\Box\bar{\chi}\bar{\eth}\chi + 16\Box^2 u\bar{u}]\phi_1(x, \theta) \tag{23}$$

where

$$(\bar{\eth})^{\dot{A}}{}_B = (\bar{\sigma}_\mu)^{\dot{A}}{}_B\partial^\mu = (\bar{\sigma}_\mu)^{\dot{\alpha}\beta}\delta^i{}_j\partial^\mu$$

and $\bar{\sigma}_\mu$ are given in the appendix. In the ϕ_2 representation, we know from equation (11*c*) that $\partial/\partial\theta_A$ is a covariant derivative and we can apply the $d/d\theta^{(4)}$ to equation (23) to obtain a superfield depending only on $\bar{\theta}$:

$$\frac{d}{d\theta^{(4)}}\phi_2(x, \theta, \bar{\theta}) \equiv \phi_2^S(x, \bar{\theta})$$

$$= F - 2i(\bar{\lambda}\bar{\eth}\bar{\theta}) + \epsilon_{XYAB}(\eth\bar{\theta})^X(\eth\bar{\theta})^Y B^{[AB]} - 8i\bar{\chi}\Box\bar{\eth}\psi + 16\Box^2 A\bar{u} \tag{24}$$

where the arrow above $\eth$ in the second term implies that the derivative acts on the λ. The superfield $\phi_1(x, \theta)$ together with $\phi_1^+(x, \bar{\theta})$ and $\phi_2^S(x, \bar{\theta})$ contain all the information we require to study the further reduction and the construction of Lagrangians.

It is a consequence of the fact that θ carries a dimensional weight, that if A has weight d, then ψ has a weight $d+\frac{1}{2}$, and so on up to F which has weight $d+2$ (or in general for SU(n) internal symmetry, F has weight $d+n$). If we now compare the two superfields which we have constructed independent of θ, namely $\phi_1^+(x, \bar{\theta})$ from equation (21) and $\phi_2^S(x, \bar{\theta})$ from equation (24), we see that the first term of ϕ_1^+ has weight d whilst the first term of ϕ_2^S has weight $d+2$, and it is possible to use the d'Alembertian to write an invariant constraint

$$\phi_2^S = c\Box\phi_1^+ \tag{25}$$

where c is a constant. Then equating coefficients in the $\bar{\theta}$ expansion gives

$$F = c\Box A^+ \tag{26a}$$

$$-2\mathrm{i}(\lambda\overleftarrow{\eth})_{\dot{A}} = c\Box\bar{\psi}_{\dot{A}} \tag{26b}$$

$$4\eth^X{}_{\dot{A}}\eth^Y{}_{\dot{B}}\tilde{B}_{[XY]} = \tfrac{1}{2}c\Box B^+_{[\dot{B}\dot{A}]} \tag{26c}$$

$$-8\mathrm{i}\Box(\bar{\eth}\psi)^{\dot{A}} = c\Box\bar{\lambda}^{\dot{A}} \tag{26d}$$

$$16\Box^2 A = c\Box F^+ \tag{26e}$$

where

$$\tilde{B}_{[XY]} = \tfrac{1}{4}\epsilon_{XYAB}B^{[AB]}.$$

At this point, it is worthwhile replacing $B^{[AB]}$ by its components labelled with SU(2) and Lorentz group indices. An SU(4) antisymmetric tensor with two indices contains a spin-one isoscalar and a spin-zero isovector, thus

$$B^{[AB]} = D_P(\tau_P g)^{ij}\epsilon_{\alpha\beta} + \tfrac{1}{4}a_{\mu\nu}(\sigma^{\mu\nu}\epsilon)_{\alpha\beta}g^{ij}. \tag{27a}$$

The tensor structure is described in the appendix. D_P is an isovector and $a_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) the spin-one isoscalar, which obeys $a_{\mu\nu} = f_{\mu\nu} - \frac{1}{2}\mathrm{i}\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}$ with $f_{\mu\nu}$ a real antisymmetric Lorentz tensor (Dondi and Sohnius 1974). Equation (27a) implies

$$\tilde{B}_{[AB]} = -\tfrac{1}{2}D_P(g\tau_P)_{ij}\epsilon^{\alpha\beta} + \tfrac{1}{8}a_{\mu\nu}(\epsilon\sigma^{\mu\nu})^{\alpha\beta}g_{ij} \tag{27b}$$

and

$$B^+_{[\dot{B}\dot{A}]} = D_P^+(g\tau_P)_{ji}\epsilon_{\dot{\beta}\dot{\alpha}} + \tfrac{1}{4}a^+_{\mu\nu}(\epsilon\bar{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\alpha}}g_{ji}. \tag{27c}$$

In terms of the fields D_P and $a_{\mu\nu}$, we find that the constraint equation (26c) is

$$\Box D_P = -\tfrac{1}{4}c\Box D_P^+ \tag{28a}$$

and

$$\partial^\mu\partial_\rho a^{\rho\nu} - \partial^\nu\partial_\rho a^{\rho\mu} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}\partial^\kappa\partial_\rho a_{\kappa\sigma} = -\tfrac{1}{4}c\Box a^{\mu\nu +}. \tag{28b}$$

From equations (26a) and (26e) it is obvious that $|c| = 4$, and choosing the phase such that $c = -4$, to give simple expressions in equation (28), we find that solutions to equations (26) and (28) which are consistent with the superfield transformations are:

$$F = -4\Box A^+ \tag{29a}$$

$$\lambda_D = -2\mathrm{i}(\bar{\psi}\overleftarrow{\bar{\eth}})_D \tag{29b}$$

$$D_P = D_P^+ \tag{29c}$$

$$f_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{29d}$$

and the number of constituent fields has been further reduced.

It is now not difficult to see that the coefficient of u in the product $\phi_1^2(x, \theta)$, which in general is given from equation (15b) by

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\phi_1^2(x,\theta) = 2AF + 2\psi^A\lambda_A + \tilde{B}_{[AB]}B^{[AB]}, \tag{30}$$

leads to an expression, when we use the constraint conditions of equation (29), which is an ideal candidate for a free massless Lagrangian, namely

$$\mathscr{L} \propto -8A\Box A^+ - 4\mathrm{i}\bar{\psi}\eth\psi + 2D_{1P}^2 - f_{\mu\nu}f^{\mu\nu}. \tag{31}$$

The integral of $\mathscr{L}$ is thus an invariant action for a complex-scalar–isoscalar, a real-vector–isoscalar, and a Weyl-spinor–isospinor. D_{1P} (the real part of D_P) is an auxiliary field and does not propagate. As an alternative we see that the general expression

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\phi_1\Box\phi_1 = 2F\Box A - 2\lambda_A\Box\psi^A + \tilde{B}_{[AB]}\Box B^{[AB]} + \text{divergence terms} \tag{32}$$

leads, for a superfield obeying the further conditions of equation (29), to

$$\mathscr{L} \propto -\tfrac{1}{2}FF^+ + \mathrm{i}\lambda\eth\bar{\lambda} + 2D_{1P}\Box D_{1P} + 2\partial_\alpha f^{\alpha\beta}\partial^\rho f_{\rho\beta} \tag{33}$$

where F is now the auxiliary field, and the peculiar term in $f_{\alpha\beta}$ in fact describes a massless scalar field†. Thus, we find that the possibility of applying the extra constraint of equation (29) leads to two free Lagrangians describing different sets of fields. A study of the rest-frame algebra shows that the simplest massive supermultiplet has more components than either of the multiplets described by the Lagrangians of equations (31) and (33) which are therefore inherently massless.

To construct a massive Lagrangian we need the complete superfield of equation (15b). However, in order to allow for the reduction just given, we redefine the superfield as

$$\phi_1(x,\theta) = A + \theta_A\psi^A + \tfrac{1}{2}\theta_A\theta_B B^{[AB]} - \chi^A(\varphi_A + 2\mathrm{i}(\bar{\psi}\eth)_A) + u(F' + 4\Box A^+), \tag{34}$$

then setting $F' = 0$, $\varphi_A = 0$ etc leads to a reduced multiplet. Further, the superfield given by

$$\phi^{\mathrm{R}} \equiv \phi_1 - \frac{1}{4\Box}(\phi_2^{\mathrm{S}})^+ \tag{35}$$

which automatically satisfies the constraint equation (25) is a reduced superfield independent of A and ψ^A. Now, in order to find the free massive Lagrangian, we consider possible bilinear combinations of superfields, as was done in the simple model without internal symmetry, and find that both

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(4)}}\phi_1^+\phi_2 \equiv \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(4)}}\phi_1^+\phi_2^{\mathrm{S}} \tag{36}$$

† We disagree with Firth and Jenkins (1975) over the particle content of the two Lagrangians given by equations (31) and (33).

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\phi_1\Box\phi_1+\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(4)}}\phi_1^+\Box\phi_1^+ \tag{37}$$

have dimensional weight 4, when A has $d = 0$, and are thus candidates for 'kinetic terms', whilst

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\phi_1^2+\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(4)}}\phi_1^{+2} \tag{38}$$

is of dimension 2 and gives a 'mass-like term'. In fact a suitable action is

$$A = \int \mathrm{d}^4x\frac{\mathrm{d}}{\mathrm{d}\theta^{(4)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(4)}}[\phi_1{}^+\phi_2-2\phi_1(\Box+2m^2)\phi_1\bar{u}(\bar{\theta})-2\phi_1^+(\Box+2m^2)\phi_1^+u(\theta)] \tag{39}$$

which in terms of the individual fields is

$$\begin{aligned}A = \int \mathrm{d}^4x[(F'F'^+ &- 2\mathrm{i}\varphi\eth\bar{\varphi}+16\partial_\mu D_{1P}\partial^\mu D_{1P}-16\partial_\alpha f^{\alpha\beta}\partial^\rho f_{\rho\beta}\\ &-4m^2(16A\Box A^+ +2AF'+2A^+F'^+ +8\mathrm{i}\bar{\psi}\eth\psi-2\varphi\psi+2\bar{\varphi}\bar{\psi}\\ &+4D_{1P}^2-4D_{2P}^2-2f_{\mu\nu}f^{\mu\nu})].\end{aligned} \tag{40}$$

The F' and D_{2P} (the imaginary part of D_P) are auxiliary fields, while the particle content is: a scalar and pseudoscalar (real and imaginary parts of mA), an isovector (D_{1P}), a vector (described by $f_{\mu\nu}$) and a Dirac spinor–isospinor and its Hermitian conjugate (obtained by arranging the two Weyl spinor–isospinors as $(^{m\psi}_{\bar{\varphi}})$). Notice that the dimensional parameter m has been used to produce fields with canonical dimension. The redefinition of $\phi_1(x,\theta)$ according to equation (34) to take into account the possible further constraint of equation (29) has provided a proper set of fields to construct a Lagrangian as in equation (40), without higher derivatives, and we know immediately that there is no problem from ghost states. We are of course still left with the problem of introducing interactions, and it has been shown (Capper and Leibbrandt 1975) that the simple ϕ_1^3 interaction is non-renormalizable. In fact, because of the auxiliary D_{2P} field that we are dealing essentially with a non-polynomial interaction. It remains to be seen if there are any renormalizable interactions when internal symmetry and supersymmetry are combined.

Finally, using the correspondence between differentiation and integration mentioned in § 3, we can write the action in the more aesthetic form:

$$A = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}[\phi_1^+\phi_2-2\phi_1(\Box+2m^2)\phi_1\,\delta(\bar{\theta})-2\phi_1^+(\Box+2m^2)\phi_1^+\,\delta(\theta)] \tag{41}$$

where we have replaced $\mathrm{d}/\mathrm{d}\theta^{(4)}$, $\mathrm{d}/\mathrm{d}\bar{\theta}^{(4)}$, $\bar{u}(\bar{\theta})$ and $u(\theta)$ in equation (39) by $\mathrm{d}^4\theta$, $\mathrm{d}^4\bar{\theta}$, $\delta(\bar{\theta})$ and $\delta(\theta)$ respectively which puts x_μ, θ and $\bar{\theta}$ on the same footing. This is the basis of the functional approach to superfields, by which superfield equations of motion and propagators etc can be obtained by functional differentiation of the action with respect to the superfields (Fujikawa and Lang 1975).

5. Conclusion

The aim of this paper has been to describe a simple and compact way of generalizing the superfield techniques of supersymmetry to cases including SU(n) as an internal symmetry group. The construction of a Lagrangian model with SU(2) as the internal symmetry is dealt with in detail, and we have shown that it is necessary to take into account the possibility of further constraints, to give a set of fields for which a Lagrangian can be found, manifestly free of ghost states. The functional methods, which have been put to good use in the case of no internal symmetry, can also be used in this model.

Although we have concentrated on a particular superfield independent of $\bar{\theta}$, we envisage no difficulty in using the SU($2n$) notation for any type of superfield.

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Appendix

The notation we use for SL(2, C) is based on the Weyl formulation. The matrices $(\sigma_\mu)_{\alpha\dot{\beta}} \equiv (1, \sigma_i)$, where σ_i are the Pauli matrices, and $(\bar{\sigma}_\mu)^{\dot{\alpha}\beta} \equiv (1, -\sigma_i)$. We define a two-dimensional ϵ tensor

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta}$$

which is different to the SU(n) ϵ tensor used in the bulk of the formulation but allows for matrix notation to be used more conveniently. The two sets of σ matrices are related by

$$(\bar{\sigma}_\mu)^{\dot{\beta}\alpha} = -(\epsilon\sigma_\mu\epsilon)^{\alpha\dot{\beta}}.$$

The $(\sigma_{\mu\nu})$ and $(\bar{\sigma}_{\mu\nu})$ are given in terms of these matrices by

$$(\sigma_{\mu\nu})_\alpha{}^\beta = \tfrac{1}{2}\mathrm{i}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\beta$$

and

$$(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \tfrac{1}{2}\mathrm{i}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}}$$

which can be considered simply as covariant rearrangements of the basic Pauli matrices. From these we can form symmetric and antisymmetric matrices $(\sigma_{\mu\nu}\epsilon)_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ with similar combinations for contravariant indices and dotted indices.

The SU(2) matrices g_{ij} and g^{ij} correspond to the two-dimensional $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ respectively, and in SU(2) we have the symmetric and antisymmetric combinations $(g\tau_P)_{ij}$ and g_{ij} and so on. Then the product of basic spinors can be projected out using combinations of these objects to give expressions like equation (27).

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